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# A note on equalities in group algebras and character values

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## 1 Introduction

In this report I reproduce my 20 minutes talk given at Kyoto RIMS on March 9, 1998.

Let  $G$  and  $H$  be finite groups of order  $n$ . A mapping  $f$  from  $G$  into  $H$  is called a planar function of degree  $n$  if, for each element  $\tau \in H$  and  $u \in G^* = G - \{1\}$ , there exists exactly one  $x \in G$  such that  $f(ux)f(x)^{-1} = \tau$ . In [2] Hiramine has shown that if both  $G$  and  $H$  are abelian groups of order  $3p$  with  $p(\geq 5)$  a prime, then there exists no planar function from  $G$  into  $H$ . To prove this he has established two results on character values. Their proofs are slightly complicated. In this note we shall give short proofs.

We follow the notation and terminology of [2].

## 2 Planar Functions and Equations in Group Algebras

Let  $G$  and  $H$  be finite groups of order  $n$ . Throughout this article elements of  $G$  will be denoted by small Roman letters and elements of  $H$  by small Greek letters.

Let  $f$  be a mapping from  $G$  into  $H$  and  $S_\alpha = \{x \in G | f(x) = \alpha\}$ ,  $\alpha \in H$ . If  $S_\alpha \neq \emptyset$ , we set  $\hat{S}_\alpha = \sum_{x \in S_\alpha} x \in C[G]$  and  $\hat{S}_\alpha^{-1} = \sum_{x \in S_\alpha} x^{-1} \in C[G]$ ,

otherwise  $\hat{S}_\alpha = \hat{S}_\alpha^{-1} = 0$ , where  $C[G]$  is the group algebra of  $G$  over the complex number field  $C$ . Let  $G_0 = G \times H$  be the direct product of groups  $G, H$ .

To prove the results we need two propositions.

The following is Proposition 2.1 [2].

**Proposition 1** *The following are equivalent.*

(i) *The function  $f$  is planar.*

(ii) *In the group algebra  $C[G]$  of  $G$ ,*

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha} \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_{\alpha\tau}^{-1} \hat{S}_\alpha = \begin{cases} \hat{G} + n - 1 & \text{if } \tau = 1, \\ \hat{G} - 1 & \text{otherwise.} \end{cases}$$

REMARK 1. If  $\tau \neq 1$ , then it follows from the equation in (ii) of the proposition above that in the group algebra  $C[G_0]$  of  $G_0$ ,

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha} \tau \alpha \hat{S}_\alpha^{-1} \alpha^{-1} = (\hat{G} - 1)\tau.$$

We prove the following

**Proposition 2** *We have in  $C[G_0]$ ,*

$$\left( \sum_{\alpha \in H} \hat{S}_\alpha \alpha \right) \left( \sum_{\beta \in H} \hat{S}_\beta^{-1} \beta^{-1} \right) = \hat{G} + n - 1 + \sum_{\tau \in H, \tau \neq 1} (\hat{G} - 1)\tau.$$

In order to prove proposition 2 we need the Remark 1 above and an easy

**Lemma 1**

$$\left( \sum_{\alpha \in H} \alpha \right) \left( \sum_{\beta \in H} \beta^{-1} \right) = \sum_{\tau \in H} \sum_{\beta \in H} (\beta\tau) \beta^{-1}.$$

### 3 Proofs of Hiramine' Results

We start with the following well-known facts about character theory. These facts play important parts in the proofs of his results.

**Fact 1** Let  $G$  be an abelian group and  $\chi$  an arbitrary (linear) character of  $G$ . Then  $\chi$  is a homomorphism from  $G$  into  $C^* = C - \{0\}$ . So we can extend this homomorphism  $\chi$  to an algebra homomorphism  $\bar{\chi}$  from  $C[G]$  into  $C$ .

**Fact 2** Let  $H_1, H_2$  be finite groups and  $G_1$  the direct product of  $H_1, H_2$ . Then all irreducible characters of  $G_1$  are obtained as follows. Let  $\chi_0, \dots, \chi_{s-1}$  be the irreducible characters of  $H_1$ ,  $\rho_0, \dots, \rho_{t-1}$  the irreducible characters of  $H_2$ . Then  $G_1$  has exactly  $st$  irreducible characters  $\Psi_{ij}$  ( $0 \leq i \leq s-1, 0 \leq j \leq t-1$ ), satisfying  $\Psi_{ij}(h_1 h_2) = \chi_i(h_1) \rho_j(h_2)$ , where  $h_1 \in H_1, h_2 \in H_2$ .

PROOF. See [1, p.54].

REMARK 2. In Fact 2 if both  $\chi_i$  and  $\rho_j$  are linear characters, then  $\Psi_{ij}$  is a homomorphism from  $G_1$  to  $C^*$ . As in Fact 1, we have an algebra homomorphism  $\bar{\Psi}_{ij}$  from  $C[G_1]$  into  $C$  which is an extension of  $\Psi_{ij}$ .

In the remainder of this section we assume that  $f$  is a planar function and that  $G$  is an abelian group of order  $n$ . Let  $\chi_0 (= 1_G), \dots, \chi_{n-1}$  be the irreducible (linear) characters of  $G$ , where  $1_G$  denote the principal character of  $G$ . We set

$$d_i^{(\alpha)} = \begin{cases} \sum_{x \in S_\alpha} \chi_i(x) & \text{if } S_\alpha \neq \emptyset, \\ 0 & \text{if } S_\alpha = \emptyset \end{cases}$$

for each  $0 \leq i \leq n-1$  and for each  $\alpha \in H$ . Now we state Hiramine' results [2] and give our proof to Result 2.

**Result 1** The following hold

(i)  $d_0^{(\alpha)} = |S_\alpha|$  and

$$\sum_{\alpha \in H} d_0^{(\tau\alpha)} d_0^{(\alpha)} = \sum_{\alpha \in H} d_0^{(\alpha\tau)} d_0^{(\alpha)} = \begin{cases} 2n-1 & \text{if } \tau = 1, \\ n-1 & \text{otherwise.} \end{cases}$$

(ii) For  $i \neq 0$ ,

$$\sum_{\alpha \in H} d_i^{(\tau\alpha)} \overline{d_i^{(\alpha)}} = \sum_{\alpha \in H} \overline{d_i^{(\alpha\tau)}} d_i^{(\alpha)} = \begin{cases} n-1 & \text{if } \tau = 1, \\ -1 & \text{otherwise.} \end{cases}$$

(Here  $\bar{d}$  denotes the complex conjugate of  $d \in C$ .)

PROOF. We omit our proof of this result.

**Result 2** *With the same notation and assumption as in Result 1, suppose that  $H$  is abelian and let  $\rho_0(= 1_H), \dots, \rho_{n-1}$  be the irreducible characters of  $H$ . Set  $z_{ij} = \sum_{\alpha \in H} d_i^{(\alpha)} \rho_j(\alpha)$ . Then,*

(i)  $z_{0,0} = n$ , and  $z_{i,0} = 0 (i \neq 0)$

(ii) For  $j \neq 0$ ,  $z_{ij} \overline{z_{ij}} = n$ .

PROOF. Since  $\chi_i$  and  $\rho_j$  are linear, from Remark 2 we see that  $\overline{\Psi}_{ij}$  is an algebra homomorphism from  $C[G_0]$  into  $C$ . First we shall prove (ii). We apply  $\overline{\Psi}_{ij}$  ( $j \neq 0$ ) to the equation in Proposition 2. Then we get (ii). We have proved (ii). Next we shall prove (i). Similarly by using  $\overline{\Psi}_{i0}$  we can prove (i). This completes the proof of Result 2.

REMARK 3. Nakagawa[3] has proved Result 2 by using Gaussian sums.

## References

- [1] L. Dornhoff: *Group representation theory, Part A*, Marcel Dekker, New York, 1971.
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- [3] N. Nakagawa: *Left cyclic planar functions of degree  $p^n$* , Utilitas Math. in press.